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**QUANTUM COMPUTER WITH ATOMIC LOGICAL
QUBITS ENCODED ON MACROSCOPIC THREE-LEVEL
SYSTEMS IN COMMON QUANTUM ELECTRODYNAMIC
CAVITY**

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ABSTRACT. We propose an effective realization of the universal set of elementary quantum gates in solid state quantum computer based on macroscopic (or mesoscopic) resonance systems – multi-atomic coherent ensembles, squids or quantum dots in quantum electrodynamic cavity. We exploit an encoding of logical qubits by the pairs of the macroscopic two- or three-level atoms that is working in a Hilbert subspace of all states inherent to these atomic systems. In this subspace, logical single qubit gates are realized by the controlled reversible transfer of single atomic excitation in the pair via the exchange of virtual photons and by the frequency shift of one of the atomic ensembles in a pair. In the case of two-level systems, the logical two-qubit gates are performed by the controlling of Lamb shift magnitude in one atomic ensemble, allowing/blocking the excitation transfer in a pair, respectively, that is controlled by the third atomic system of another pair. When using three-level systems, we describe the NOT-gate in the atomic pair controlled by the transfer of working atomic excitation to the additional third level caused by direct impact of the control pair excitation. Finally, we discuss advantages of the proposed physical system for accelerated computation of some useful quantum gates.

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1. INTRODUCTION

Various physical systems using single natural or artificial atoms, ions, molecules etc. have been proposed during two last decades for the construction of a quantum computer [1, 2, 3, 4]. Creation of a quantum computer with a large number of qubits is a huge problem on the known systems, first of all, because of too strong decoherence of the qubits. That makes the search for the new physical systems and relevant experimental approaches still actual. One of the promising approaches is using natural atoms (ions, molecules, ...) with long coherence time.

Recently, new physical realization of a quantum computer based on the *multi-atomic coherent* (MaC) ensembles has been proposed for encoding of separate qubits [5, 6]. MaC ensembles yield huge amplification of dipole moment on the resonance transition that leads to essential acceleration of the quantum information processing rate. However, transitions to the excess states in a MaC ensemble should be blocked in order to realize effective two-level systems providing an ideal encoding of qubits. Dipole-dipole interaction is intensively discussed for the blockade of excess quantum states [7]. But the mechanism of dipole blockade is limited by the radius of dipole-dipole interaction and can suffer from the decoherence problems arising because of the strong dependence of dipole-dipole interaction on spatial distance between the interacting atoms. Besides, another blockade mechanism was recently proposed based on the dependence of Raman transition frequency on the number of photons in signal field [8] that still remains rather complicated for experimental realization. We have also proposed a new decoherence free blockade mechanism based on the Lamb shift in quantum electrodynamics cavity with additional micro-resonators [9, 10, 11, 12]. Rapid development of micro-resonators physics and technology [13, 14, 15] makes this blockade mechanism very promising though not so simple for experimental realization.

In this paper, we demonstrate how both single qubit and two qubit gates with logical encoding of qubits [16, 17, 18, 19, 20, 21, 22] can be realized in natural way on MaC ensembles in the QED cavity without additional micro-cavities by using only some definite operations of swapping by excitations between MaC ensembles. Here, we use the encoding of qubits by pairs of MaC ensembles. In this context the operation of excitation swapping between such two ensembles corresponds to the NOT gate and swapping operation controlled by photon from control qubit corresponds to CNOT operation. We consider physical realization of such quantum gates and show that they can be used for creation of the quantum computer satisfying the necessary DiVincenzo criteria [23].

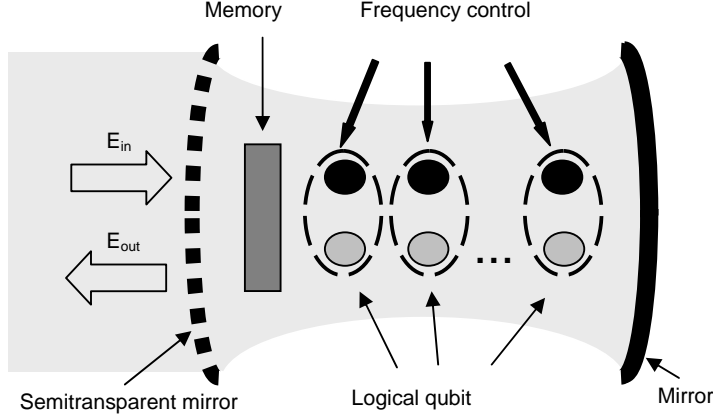


FIGURE 1. Scheme of quantum computer based on the atomic logical qubits encoded on macroscopic multi-atomic systems in common quantum electrodynamic cavity.

2. EXCITATION SWAPPING ON THREE LEVEL SYSTEMS

Let's consider a set of three-level MaC ensembles (nodes) posted in the common electrodynamic cavity (Fig. 1). Equalizing of resonance transitions frequencies by the external control electric/magnetic fields leads to fast excitation swapping between the atoms of nodes by the interaction via the field of virtual photons that can be described by an effective atomic Hamiltonian [24]. In order to find this interaction for the large number of atoms, we start from the initial Hamiltonian $H = H_0 + H_1$, where $H_0 = H_d + H_f$ is the main Hamiltonian and $H_1 = H_{d-f}$ is the perturbation Hamiltonian. Here, $H_d = \sum_{m=1}^2 \sum_{\mu=1}^3 \sum_{j_m=1}^{N_m} \varepsilon_m^\mu S_{\mu\mu}^{j_m}$ is Hamiltonian of atoms in nodes 1 and 2 in the terms of operator generators $S_{\mu\mu}^{j_m}$ of SU(3) group, where N_m is a number of atoms in m -th node, ε_m^μ is an energy of level μ in the m -th node, and $H_f = \sum_{\alpha}^{1,2,3} \hbar \omega_{k_\alpha} a_{k_\alpha}^+ a_{k_\alpha}$ is Hamiltonian of photons, where ω_{k_α} is the frequency of photons with wave vector k_α , $a_{k_\alpha}^+$ and a_{k_α} are operators of creation and annihilation for photons in modes 1, 2 and 3 (Fig. 1). For the interaction of photons with atoms in nodes 1 and 2 $H_{r-a} = H_{r-a}^{(1)} + H_{r-a}^{(2)} = H_{21} + H_{31} + H_{32}$ we have the following expressions on corresponding atomic transitions:

$$(1) \quad H_{21} = \sum_m^{1,2} \sum_{j_m} \left(g_{21}^{k_1} e^{i\vec{k}_1 \vec{r}_{j_m}} S_{21}^{j_m} a_{k_1} + g_{21}^{k_1*} e^{-i\vec{k}_1 \vec{r}_{j_m}} S_{12}^{j_m} a_{k_1}^+ \right),$$

$$(2) \quad H_{32} = \sum_m \sum_{j_m}^{1,2} \left(g_{32}^{k_2} e^{i\vec{k}_2 \vec{r}_{j_m}} S_{32}^{j_m} a_{k_2} + g_{32}^{k_2*} e^{-i\vec{k}_2 \vec{r}_{j_m}} S_{23}^{j_m} a_{k_2}^+ \right),$$

$$(3) \quad H_{31} = \sum_m \sum_{j_m}^{1,2} \left(g_{31}^{k_3} e^{i\vec{k}_3 \vec{r}_{j_m}} S_{31}^{j_m} a_{k_3} + g_{31}^{k_3*} e^{-i\vec{k}_3 \vec{r}_{j_m}} S_{13}^{j_m} a_{k_3}^+ \right),$$

where $g_{\mu\nu}^{(k_\alpha)}$ is the interaction constant for transition from level μ to level ν , $S_{\mu\nu}^{j_m}$ are the operators of the transitions between the levels $\mu \leftrightarrow \nu$, \vec{r}_{j_m} is a radius vector of atoms j_m in nodes $m = 1, 2$.

In the Appendix A, we derive an effective Hamiltonian describing inter-atomic interaction via the virtual photons of QED cavity. The interaction opens the way for arbitrary quantum manipulation of the atomic ensembles.

In the case when $\Delta_m^{(1)} \ll \Delta_m^{(2)}, \Delta_m^{(3)}$, we can leave in Hamiltonian (35) only the terms concerning levels 1 and 2 and write it at $\Delta = \Delta_1^{(1)} = \Delta_2^{(1)}$ as following:

$$(4) \quad H_s = \sum_m \sum_{j_m}^{1,2} \left(\varepsilon_m^{(1)} S_{11}^{j_m} + \varepsilon_m^{(2)} S_{22}^{j_m} \right) + \Omega_\sigma \sum_{m=1,2} \sum_{i_m j_m} e^{i\vec{k}_1 \vec{r}_{i_m j_m}} S_{21}^{i_m} S_{12}^{j_m} \\ + \Omega_\sigma \sum_{j_1 j_2} \left(e^{i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{21}^{j_1} S_{12}^{j_2} + e^{-i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{12}^{j_1} S_{21}^{j_2} \right),$$

where $\Omega_\sigma = |g_{21}^{k_3}|^2 / \Delta$. Here, the first term describes the atomic energies in the nodes $m = 1$ and 2, the second term is the energy of atomic excitation swapping inside the nodes, the third term is the energy of atomic excitation swapping by virtual photons between various nodes.

Let's introduce a finite set of collective basic states of all possible Hilbert states of the multi-atomic systems in the two nodes: $|\psi\rangle_1 = |0\rangle_1 |0\rangle_2$, $|\psi\rangle_2 = |1\rangle_1 |0\rangle_2$, $|\psi\rangle_3 = |0\rangle_1 |1\rangle_2$, $|\psi\rangle_4 = |1\rangle_1 |1\rangle_2$, $|\psi\rangle_5 = 1/\sqrt{2} \{ |2\rangle_1 |0\rangle_2 + |0\rangle_1 |2\rangle_2 \}$, $|\psi\rangle_6 = 1/\sqrt{2} \{ |2\rangle_1 |0\rangle_2 - |0\rangle_1 |2\rangle_2 \}$, where $|0\rangle_{1,2} = \prod_{j_{1,2}}^{N_{1,2}} |g\rangle_{j_{1,2}}$, $|1\rangle_{1,2} = (1/\sqrt{N_{1,2}}) \sum_{j_{1,2}=1}^{N_{1,2}} |e\rangle_{j_{1,2}} \prod_{l_{1,2} \neq j_{1,2}}^{N_{1,2}} |g\rangle_{l_{1,2}}$ and $|2\rangle_{1,2} = \sqrt{2/N_{1,2}(N_{1,2}-1)} \cdot \sum_{j_{1,2} > f_{1,2}}^{N_{1,2}} |e\rangle_{j_{1,2}} |e\rangle_{f_{1,2}} \prod_{l_{1,2} \neq j_{1,2}, f_{1,2}}^{N_{1,2}} |g\rangle_{l_{1,2}}$. Here for convenience, the notations: $|g\rangle_{j_{1,2}}$ and $|e\rangle_{j_{1,2}}$ are introduced for the ground and the first excited states of atom $j_{1,2}$. It is worth noting that the atomic states $|\psi\rangle_2$ and $|\psi\rangle_3$ can be prepared by the interaction with single photon states of the external photon sources. Using these collective states, we find with the initial state of two qubits of the atoms in two nodes

$$(5) \quad |\Psi_{in}(0)\rangle = \{ \alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1 \} \{ \alpha_2 |0\rangle_2 + \beta_2 |1\rangle_2 \},$$

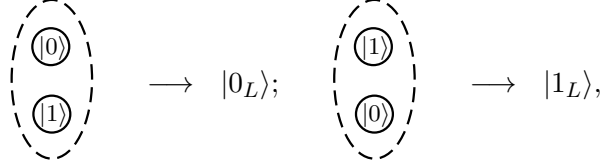


FIGURE 2. Pairwise qubit encoding. Small circles denote the processing nodes in the indicated quantum states.

the following unitary evolution of atomic systems under the interaction part of Hamiltonian (4):

$$\begin{aligned}
 |\Psi(t)\rangle = & \alpha_1 \alpha_2 |\psi\rangle_1 + \\
 (6) \quad & + \exp(-i\Omega_\sigma Nt) \{ \beta_1 \alpha_2 [\cos(\Omega_\sigma Nt) |\psi\rangle_2 - i \sin(\Omega_\sigma Nt) |\psi\rangle_3] + \\
 & + \alpha_1 \beta_2 [\cos(\Omega_\sigma Nt) |\psi\rangle_3 - i \sin(\Omega_\sigma Nt) |\psi\rangle_2] \} + \\
 & + \exp(-i2\Omega_\sigma Nt) \beta_1 \beta_2 [\cos(2\Omega_\sigma Nt) |\psi\rangle_4 - i \sin(2\Omega_\sigma Nt) |\psi\rangle_5],
 \end{aligned}$$

where we have assumed that the quantity of atoms is sufficiently large $N_1 = N_2 = N \gg 1$.

The solution (6) demonstrates availability of two coherent oscillations with frequency $\Omega_\sigma N$ for the first pair of states $|\psi\rangle_2 \leftrightarrow |\psi\rangle_3$ and with doubled frequency $2\Omega_\sigma N$ for the second pair $|\psi\rangle_4 \leftrightarrow |\psi\rangle_5$. Oscillations are strongly accelerated (N and $2N$ times) comparing to the case of quantum oscillations of two coupled two-level atoms. We have performed a special numeric analysis of the effective Hamiltonian approach (4) which has demonstrated that large enough spectral detuning $\Delta > 30\sqrt{N} |g_{21}^{k3}|$ will provide an error less than 0.001 for the quantum dynamics presented in Eq. (6).

3. QUANTUM GATES

3.1. Single-qubit gates. Here, we introduce an encoding of logical qubits by the pairs of multiatomic systems [16, 17, 18, 19, 20, 21, 22] (see Fig. 2).

The quantum evolution (6) describes the operation that we call the *Excitation Transfer*, denoted by $ET(\theta)$, where $\theta = \Omega_\sigma Nt$ is an arbitrary angle. This operation acts on basis states $|0\rangle|1\rangle$ and $|1\rangle|0\rangle$ as following:

$$\begin{aligned}
 (7) \quad & |0\rangle|1\rangle \longrightarrow \cos \frac{\theta}{2} |0\rangle|1\rangle - i \sin \frac{\theta}{2} |1\rangle|0\rangle \\
 & |1\rangle|0\rangle \longrightarrow -i \sin \frac{\theta}{2} |0\rangle|1\rangle + \cos \frac{\theta}{2} |1\rangle|0\rangle.
 \end{aligned}$$

Thus, for a logical qubit with basis states $|0\rangle_L = |0\rangle|1\rangle$ and $|1\rangle_L = |1\rangle|0\rangle$, the $ET(\theta)$ gate corresponds to the rotation $R_x(\theta)$ around the x axis of the Bloch sphere.

So, a quantum computer on multi-atomic systems in the common quantum electrodynamic cavity (Fig. 1) can perform a single qubit rotation in the following way: first, quantum memory is put into the resonance with designated processing node by equalizing frequencies of this node and the memory, then the excitation is transferred from the memory to the processing node by a real photon as described in the paper [24]. Next the memory is decoupled from the node and the node is equalized in frequency with another processing node of logical qubit so the ET (θ) operation takes place between the two nodes. Finally, the excitation is returned to the memory in the same way it was transferred from there.

The proposed architecture also allows us to perform the gate PHASE(χ), described as following

$$(8) \quad \begin{aligned} |0\rangle|1\rangle &\longrightarrow e^{-i\chi/2}|0\rangle|1\rangle \\ |1\rangle|0\rangle &\longrightarrow e^{i\chi/2}|1\rangle|0\rangle \end{aligned} ,$$

where the phases $\chi = \delta\omega\tau$ and $\delta\omega$ can be controlled by differing external magnetic (or electric) fields on the spatially distinct nodes that provides, respectively, different Zeeman (or Stark) frequency shifts for the two-level atoms localized in the nodes [9].

When using pairwise encoding of qubits PHASE(χ) turns a logical qubit around the axis \hat{z} , thus corresponding to the logical $R_{\hat{z}}(\chi)$ operator.

That is, ET(θ) and PHASE(χ) act like orthogonal rotations of the logical qubits, thus allowing to perform an arbitrary rotation using these elementary operations.

We note that the described realization of single-qubit gates doesn't use any resonant electromagnetic fields that prevents the quantum operations of the additional excess field noises.

3.2. Two-qubit gates. In order to perform universal quantum computation in the Hilbert subspace corresponding to the pairwise qubit encoding we need an entangling two-qubit gate such as CNOT gate. In our architecture we will use a Controlled-ET(π) gate (for short, C(ET)).

Since the ET(π) operation turns $|01\rangle$ into $|10\rangle$ and backwards (up to the global phase factor $-i$), thus acting on a pair like the NOT gate, C(ET) implements the logical CNOT gate up to the relative phase shift, which can be made global by the PHASE($\pi/2$) operation applied to the control qubit. (see Fig. 3).

Formally:

$$(9) \quad \begin{aligned} &(\text{PHASE}(\pi/2) \otimes I) \cdot \text{C(ET)} \cdot (\alpha_1|0_L\rangle|0_L\rangle + \alpha_2|0_L\rangle|1_L\rangle + \alpha_3|1_L\rangle|0_L\rangle + \\ &+ \alpha_4|1_L\rangle|1_L\rangle) = \\ &(\text{PHASE}(\pi/2) \otimes I) \cdot (\alpha_1|0_L\rangle|0_L\rangle + \alpha_2|0_L\rangle|1_L\rangle - i\alpha_3|1_L\rangle|1_L\rangle - i\alpha_4|1_L\rangle|0_L\rangle) = \\ &e^{-\frac{\pi}{4}} (\alpha_1|0_L\rangle|0_L\rangle + \alpha_2|0_L\rangle|1_L\rangle + \alpha_3|1_L\rangle|1_L\rangle + \alpha_4|1_L\rangle|0_L\rangle) . \end{aligned}$$

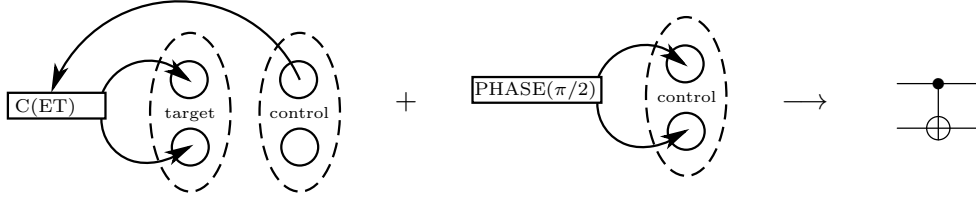


FIGURE 3. Logical CNOT gate implemented by the physical C(ET) gate coupled with PHASE ($\pi/2$).

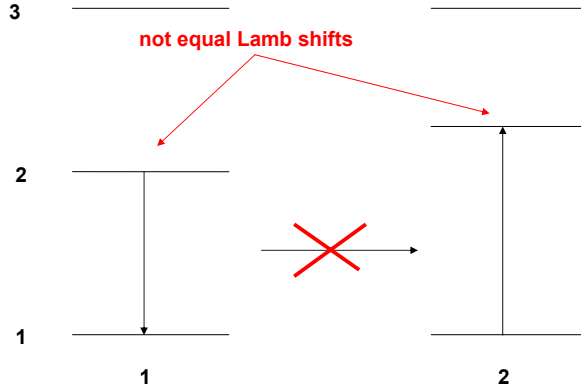


FIGURE 4. Operation Controlled-ET at not equivalent Lamb shifts. In the absence of shifts, operation ET proceeds. At the presence of not equivalent shifts under the influence of the field from control qubit operation ET is blocked.

We propose two schemes for realization of C(ET) operation on the logical qubits. The first scheme exploits not equivalent Lamb shift of target logical qubit under the influence of control qubit field (Fig. 4). In the absence of the frequency shifts, ET operation proceeds as it is described in the previous subsection. The second scheme based on quantum transistor effect uses a transition of working excitation in target logical qubit at the additional level under the direct impact of control pair excitation. This transition takes place if there are atoms of the control qubit in excited state and the transition then blocks ET operation in the targetlogical qubit. Otherwise, if all the atoms of control qubit are in the ground state, there is no such a transition and ET operation in the targetlogical qubit takes place.

Additionally, we will call a compound PHASE ($\pi/2$)-C(ET) operator a *Phased Controlled Excitation Transfer* (PCET). Since this operation acts like Controlled-SWAP for our logical encoding, we will use the C(SWAP) circuit notation for PCET (as depicted in Figures 8, 10).

3.2.1. Gates on nonequivalent Lamb shift. In order to describe blocking of ET operation at not equivalent Lamb shifts, we will take into account that if the number of real photons in the cavity $n_{k_1} \neq 0$ then we will have the following Hamiltonian instead of (4)

$$\begin{aligned}
 H_s = & \sum_m \sum_{j_m \mu} \varepsilon_m^\mu S_{\mu\mu}^{j_m} + \sum_{m=1,2} \frac{|g_{21}^{(m)}(k_1)|^2}{\Delta_1^{(m)}} \sum_{j_m} \left(S_{22}^{j_m} - S_{11}^{j_m} \right) n_{k_1} \\
 (10) \quad & + \sum_{m=1,2} \frac{|g_{21}^{(m)}(k_1)|^2}{\Delta_1^{(m)}} \sum_{i_m j_m} e^{i\vec{k}_1 \vec{r}_{i_m j_m}} S_{21}^{i_m} S_{12}^{j_m} \\
 & + \frac{1}{2} \left(\frac{1}{\Delta_1^{(1)}} + \frac{1}{\Delta_1^{(2)}} \right) \sum_{j_1 j_2} \left(g_{21}^{(1)}(k_1) g_{21}^{(2)}(k_1)^* e^{i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{21}^{j_1} S_{12}^{j_2} \right. \\
 & \left. + g_{21}^{(1)}(k_1)^* g_{21}^{(2)}(k_1) e^{-i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{12}^{j_1} S_{21}^{j_2} \right).
 \end{aligned}$$

Now, the wave function of system can be written in the following form:

$$(11) \quad \psi(t) = \sum_n \sum_{l=1}^{0,1,6} c_l^{(n)}(t) \psi_l |n\rangle,$$

where ψ_i are given in previous section, and for $n = 1$ we get

$$(12) \quad \psi(t, n = 1) = \sum_{l=1}^6 c_l^{(1)}(t) \psi_l |1\rangle.$$

By writing Schrodinger equation for the wave function (12), we can see that, in analogy to the evolution described by matrix(7), transitions between the states $|1\rangle_1 |0\rangle_2$ and $|0\rangle_1 |1\rangle_2$ proceed separately from transitions between other atomic states.

Let $\Omega_1 = \frac{|g_{k_1}^{(1)}|^2}{\hbar^2 \Delta_1}$, $\Omega_2 = \frac{|g_{k_1}^{(2)}|^2}{\hbar^2 \Delta_2}$, $\Omega_\sigma = \frac{g_{k_1}^{(1)} g_{k_1}^{(2)*}}{2\hbar^2} \left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right)$, and $\tilde{\omega}_1 = \omega_1 + 2n_{k_1} \Omega_1$, $\tilde{\omega}_2 = \omega_2 + 2n_{k_1} \Omega_2$. Then the equations for coefficients $A_2 = c_2^{(1)}$ and $A_3 = c_3^{(1)}$ can be written in the following form:

$$(13) \quad \frac{dc_2}{dt} = i \left\{ \left(\frac{N_1}{2} - 1 \right) \tilde{\omega}_1 + \frac{N_2}{2} \tilde{\omega}_2 - N_1 \Omega_1 \right\} c_2 - i \sqrt{N_1 N_2} \Omega_\sigma c_3,$$

$$(14) \quad \frac{dc_3}{dt} = i \left\{ \frac{N_1}{2} \tilde{\omega}_1 + \left(\frac{N_2}{2} - 1 \right) \tilde{\omega}_2 - N_2 \Omega_2 \right\} c_3 - i \sqrt{N_1 N_2} \Omega_\sigma c_2,$$

where N_1 and N_2 are the number of atoms in nodes 1 and 2. Solution of equations (13), (14) is:

$$(15) \quad c_3 = C_1 e^{ir_1 t} + C_2 e^{ir_2 t},$$

where C_1 and C_2 are constant coefficients,

$$(16) \quad r_{1,2} = \frac{1}{2} \left\{ (N_1 - 1) \tilde{\omega}_1 + (N_2 - 1) \tilde{\omega}_2 - N_1 \Omega_1 - N_2 \Omega_2 \right\} \pm \sqrt{(\tilde{\omega}_2 - \tilde{\omega}_1 + N_2 \Omega_2 - N_1 \Omega_1)^2 + 4N_1 N_2 \Omega_\sigma^2} \Bigg\}.$$

If $c_2(0) = 1$ and $c_3(0) = 0$ we get

$$(17) \quad C_1 = -C_2 = -\frac{\sqrt{N_1 N_2} \Omega_\sigma}{\sqrt{(\tilde{\omega}_2 - \tilde{\omega}_1 + N_2 \Omega_2 - N_1 \Omega_1)^2 + 4N_1 N_2 \Omega_\sigma^2}}.$$

When relation $\omega_2 - \omega_1 + N_2 \Omega_2 - N_1 \Omega_1 = 0$ holds we have

$$(18) \quad C_1 = -\frac{\sqrt{N_1 N_2} \Omega_\sigma}{2\sqrt{N_1 N_2 \Omega_\sigma^2 + n_{k_1}^2 (\Omega_2 - \Omega_1)^2}}.$$

If $n_{k_1}^2 (\Omega_2 - \Omega_1)^2 \gg N_1 N_2 \Omega_\sigma^2$ and $n_{k_1} = 1$ then $C_1 = -C_2 = c_3(t) \cong 0$ for any moment of time and ET operation does not proceed. In the case of $\Omega_2 \gg \Omega_1$, we have $\Omega_2^2 \gg N_1 N_2 \Omega_\sigma^2$ and at $\Delta_1 = \Delta_2$ we have $\Omega_\sigma^2 = \Omega_1 \Omega_2$ that leads to the following condition $\Omega_2 \gg N_1 N_2 \Omega_1$. The condition puts the limit on possible number of atoms in the nodes. If $n_{k_1} = 0$ then $C_1 = -\frac{1}{2}$ and $C_2 = \frac{1}{2}$. Hence,

$$(19) \quad c_2 = e^{\frac{i}{2\hbar}(E_2 + E_3)t} \cos\left(\sqrt{N_1 N_2} \Omega_\sigma t\right),$$

$$(20) \quad c_3 = -ie^{\frac{i}{2\hbar}(E_2 + E_3)t} \sin\left(\sqrt{N_1 N_2} \Omega_\sigma t\right),$$

where

$$(21) \quad E_2 = \hbar \left\{ \left(\frac{N_1}{2} - 1 \right) \omega_1 + \frac{N_2}{2} \omega_2 - N_1 \Omega_1 \right\},$$

$$(22) \quad E_3 = \hbar \left\{ \frac{N_1}{2} \omega_1 - \left(\frac{N_2}{2} - 1 \right) \omega_2 - N_2 \Omega_2 \right\}$$

are energies of corresponding states.

Thus, the presence of a photon in the cavity ($n_{k_1} = 1$) can block ET operation in the target logical qubit. If this photon arrives from the control qubit, then we have Controlled-ET operation. Equations (24,25) show that for small values of Ω_1 , gate rate is N times higher if $N_1 = N_2 = N$. But above shown

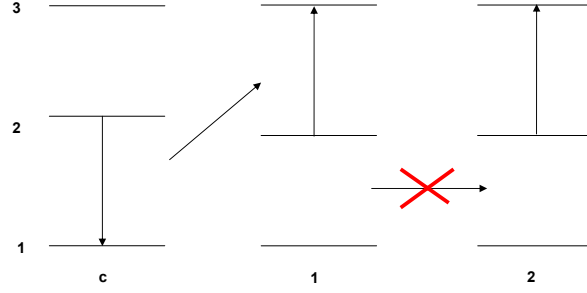


FIGURE 5. Operation Controlled-ET via quantum transistor effect. When the atom of the control logical qubit is not excited operation ET proceeds between the atoms **1** and **2** of the controlled qubit. When the atom is excited the ET process at the transition $2 \rightarrow 1$ of atom and $2 \rightarrow 3$ of atom **1** and **2** blocks ET operation between atoms **1** and **2**.

constraints put a general upper bound on the gate rate equal to Ω_2 not depending on N . So, the best choice in this case is using quantum dots with large magnitude of intrinsic dipole moment leading to large Ω_2 .

3.2.2. Gates on quantum transistor effect. Another approach to the construction of two-qubit gates on logical qubits is based on the quantum transistor effect where the ET operation between one of three-level atomic ensembles of the control qubit at lower quantum transition and both three-level atomic ensembles of the target qubit at upper transition is used (Fig. 5). Here, the frequencies of relevant transitions are put by the external driving fields in close correspondence to the frequency of the cavity and the frequencies of the target qubit atomic lower transitions are detuned far from the resonance. After termination of this process, frequencies of the target qubit atomic lower transitions are put in close correspondence to the frequency of the cavity and the frequencies of aforementioned transitions are detuned far from the resonance. As a result, if the electron of control qubit atom was in the initial moment of time in the excited state then the ET process in the target qubits does not proceed since the electrons of its atoms are on the auxiliary level 3. If, on the contrary, the electron of the control qubit atom is in the ground state at the initial moment of time then electrons of the target qubits atoms are partially on the level 2 and ET process takes place in the target qubits.

Transition frequencies are returned to the previous values after elapsing of time necessary for ET process in target qubit. As a result, depending on initial conditions electrons are transferred from the level 3 of the target qubit atoms to level 2 with the excitation of the control qubit atom in the ET process or

electron rests on level 2 of the target qubits atom 2 and operation Controlled-ET is fully executed.

Hamiltonian describing ET between one of the three-level atomic ensembles of the control qubit at the lower transition and two of the three-level atomic ensembles of the target qubits at the upper transition can be written as:

$$(23) \quad H_{ET}^{c \rightarrow 1} = \frac{1}{2} \left(\frac{1}{\Delta_1^{(c)}} + \frac{1}{\Delta_2^{(1)}} \right) \sum_{j_c j_1} \left\{ g_{32}^{(c)*} g_{21}^{(c)} e^{i(\vec{k}_1 \vec{r}_{j_c} - \vec{k}_2 \vec{r}_{j_1})} S_{21}^{j_c} S_{23}^{j_1} + g_{32}^{(1)} g_{21}^{(c)*} e^{-i(\vec{k}_1 \vec{r}_{j_c} - \vec{k}_2 \vec{r}_{j_1})} S_{12}^{j_c} S_{32}^{j_1} \right\},$$

where $\frac{1}{\Delta_0^{21}} = \frac{1}{\varepsilon_0^{(2)} - \varepsilon_0^{(1)} - \hbar\omega_{k_0}}$, $\frac{1}{\Delta_{1,2}^{32}} = \frac{1}{\varepsilon_{1,2}^{(3)} - \varepsilon_{1,2}^{(2)} - \hbar\omega_{k_0}}$. If $\frac{g_{21}^{k_0}}{\Delta_0^{21}} \gg \frac{g_{32}^{k_0}}{\Delta_{1,2}^{32}}$ we can neglect the ET process at transition $3 \rightarrow 2$ during energy transfer from control to target qubit. We can, also, block the $3 \rightarrow 2$ ET process executing this energy transfer to two nodes of the target qubit separately in time detuning frequencies of target qubit nodes from each other. In the last case, the above condition is not necessary and we can work with wider set of parameters.

Let's assume for simplicity that atomic ensemble of the control logical qubit is in the single photonic excited state

$$|1\rangle^{(A)} = N^{-1/2} \sum_{j_c} e^{i\vec{k}_1 \vec{r}_{j_c}} |0\rangle_1 \dots |1\rangle_{j_c} \dots |0\rangle_N$$

and atomic ensemble of the target logical qubit transfers as a result of the ET process to two photonic excited states

$$|2\rangle^{(m)} = N^{-1/2} \sum_{j_m} e^{i(\vec{k}_1 + \vec{k}_2) \vec{r}_{j_m}} |0\rangle_{1m} \dots |2\rangle_{j_m} \dots |0\rangle_{N_m}$$

of target qubit nodes $m = 1, 2$. Then taking into account the initial state at $t = 0$

$$(24) \quad |\psi(0)\rangle = |1\rangle^{(A)} \{ \alpha |0\rangle^{(1)} |1\rangle^{(2)} + \beta |1\rangle^{(1)} |0\rangle^{(2)} \},$$

where $|\alpha|^2 + |\beta|^2 = 1$, quantum ET-dynamics leads to the following state of atomic ensembles

$$(25) \quad |\psi(t)\rangle = |1\rangle^{(A)} \cos(\sqrt{N}\Omega t) \{ \alpha |0\rangle^{(1)} |1\rangle^{(2)} + \beta |1\rangle^{(1)} |0\rangle^{(2)} \} - i \sin(\sqrt{N}\Omega t) |0\rangle^{(A)} \{ \alpha |0\rangle^{(1)} |2\rangle^{(2)} + \beta |2\rangle^{(1)} |0\rangle^{(2)} \},$$

where

$$(26) \quad \Omega = \frac{g_{21}^{k_0} g_{32}^{k_0}}{2\hbar} \left(\frac{1}{\Delta_0^{12}} + \frac{1}{\Delta_1^{23}} \right).$$

Here, $\Delta_1^{23} = \Delta_2^{23}$ and the factor \sqrt{N} indicates a considerable acceleration of the gate rate comparing with the case of a single atom.

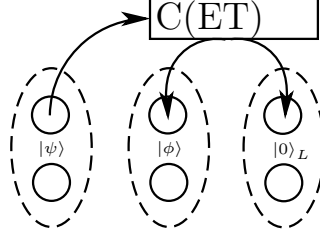


FIGURE 6. Implementing logical AND operation using C(ET).

During QT process (25,26), also, QT process between nodes of target qubit at lower transitions partially takes place. Therefore, it is more convenient practically to achieve energy transfer from atom of control qubit to atoms of target qubit one by one detuning for a while resonance between atoms of target qubit in order to avoid excess QT process.

4. IMPROVED CONSTRUCTIONS FOR USEFUL QUANTUM GATES

In this section we demonstrate interesting additional opportunities of the used atomic systems for accelerated realization of some logical operations. These opportunities will be possible if we preserve one atomic ensemble in the pure ground state thus using the states outside our logical Hilbert subspace. Since due to possible atomic interactions, the atoms could be excited in two possible states (ground state $|0\rangle$, or $|1\rangle$) this opens a possibility to efficiently construct a class of useful quantum gates.

The key idea is to use the PCET operation to compute the logical AND operation in a manner depicted in Fig. 6. It acts as following:

$$|\phi_1\phi_2\rangle|\psi_1\psi_2\rangle|0\rangle|1\rangle \longrightarrow |\phi_1\phi_2\rangle|\widetilde{\psi_1\psi_2}\rangle|\phi_1 \text{ AND } \psi_1\rangle|1\rangle.$$

It can be easily seen that the first node of the third pair will turn into the state $|1\rangle$ iff both $|\phi\rangle$ and $|\psi\rangle$ were in the state $|1\rangle$ (which shows that this operation actually performs the AND of ϕ and ψ):

$$\begin{aligned} &(\alpha_1|01\rangle|01\rangle + \alpha_2|01\rangle|10\rangle + \alpha_3|10\rangle|01\rangle + \alpha_4|10\rangle|10\rangle) \otimes |01\rangle \longrightarrow \\ &(\alpha_1|01\rangle|01\rangle + \alpha_2|01\rangle|10\rangle + \alpha_3|10\rangle|01\rangle) \otimes |01\rangle + \alpha_4|10\rangle|00\rangle|11\rangle. \end{aligned}$$

However, this operation extends the logical encoding which we have adopted for our model and we have to make sure to return the state into our Hilbert subspace after performing such “low-level” operations. In that case we don’t actually need the third qubit as a pair of nodes, all we need is a single node in the ground state $|0\rangle$.

As an example let’s consider one of the most commonly used quantum gate – the Toffoli gate (also known as CCNOT). Using the universal set of CNOT

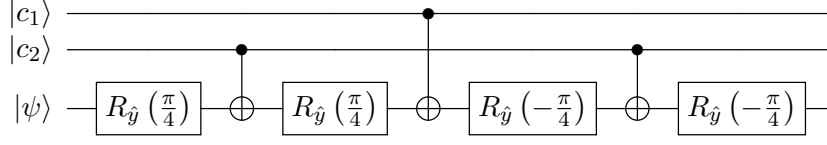


FIGURE 7. Circuit implementing Toffoli gate up to relative phase factor.

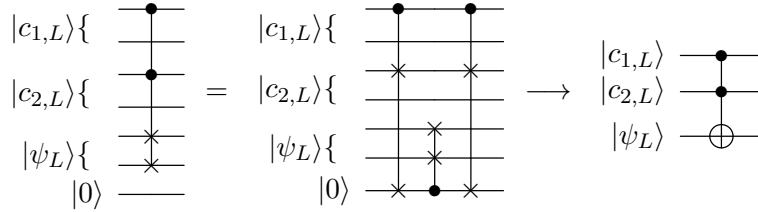


FIGURE 8. Efficient implementation of the encoded Toffoli gate using the physical PCET operations (depicted using Controlled-SWAP notation).

and single qubit gates it can be implemented up to relative phase factor using the circuit from [1] (see Fig. 7).

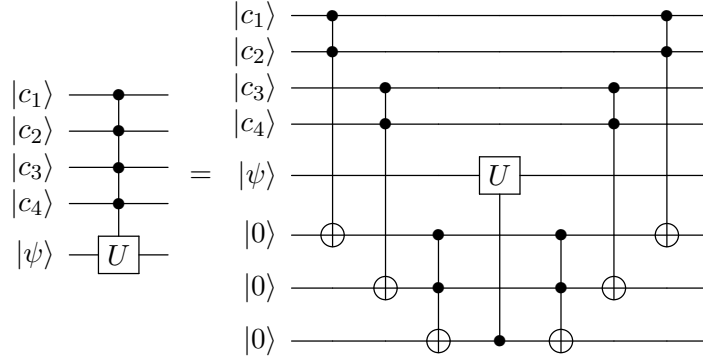
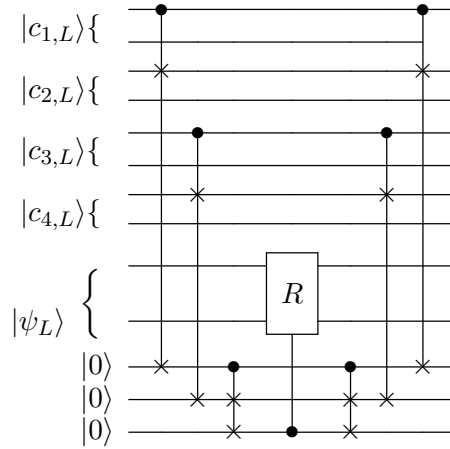
We can construct a more efficient “low-level” circuit using PCET gates and an extra processing node in the state $|0\rangle$. The circuit in Fig. 8 checks if both of control qubits (logical) are in the state $|1\rangle$ and stores the result in an auxiliary qubit (physical, single node). Afterwards, it performs the PCET of the third pair controlled by the auxiliary qubit, so the third qubit $|\psi_L\rangle$ is flipped iff both of $|c_{1,L}\rangle$ and $|c_{2,L}\rangle$ were in the state $|1_L\rangle$, which is exactly the Toffoli gate. Finally, our circuit uncomputes the first operation, returning the auxiliary qubit into state $|0\rangle$ and the whole system into the Hilbert subspace corresponding to our logical qubit encoding.

This approach can be generalized to improve constructions of the general controlled gate $C^t(U)$, defined by the following equation in [1]:

$$(27) \quad C^t(U)|c_1 c_2 \dots c_t\rangle|\psi\rangle = |c_1 c_2 \dots c_t\rangle U^{c_1 \cdot c_2 \dots c_t} |\psi\rangle.$$

The usual construction exploiting ancillary qubits in state $|0\rangle$ is the one demonstrated for $t = 4$ in Fig. 9.

Such a circuit for $C^t(U)$ requires $2(t-1)$ Toffoli gates and one controlled- U operation plus $t-1$ qubits (initially in the state $|0\rangle$) for temporary storage. As we already know each encoded Toffoli gate can be implemented using three PCET gates and additional node (“half” of the logical qubit) in the state $|0\rangle$. Of course, for non-parallel Toffoli gates we can use the same ancillary node, for it remains in the state $|0\rangle$.

FIGURE 9. Implementation of the gate U controlled by 4 qubits.FIGURE 10. Improved implementation of the gate U controlled by 4 qubits.

On the other hand, we can use the scheme of Fig. 10 for implementing the logical $C^t(U)$ gate. Here, we have made use of $2(t-1)$ PCETs (which are implemented by two elementary gates in our model) and $t-1$ ancillary processing nodes instead of full logical qubits.

5. CONCLUSION

Thus, we have introduced a logical encoding of qubits for multi-atomic systems and demonstrated its advantages for construction of universal quantum computations. Here, the logical qubits are realized by the pairs of atomic ensembles that provides a high rate of single qubit gates without excess noises on the resonant atomic frequencies. We have elaborated this approach for the construction both of single and two-qubit gates based on transfer of excitation

between the atomic ensembles via exchange of virtual photons in the QED cavity. This opens rich possibilities for quantum processing over the large number of qubits. The proposed architecture is scalable due to the free of decoherence nature of interaction between the multi-atomic ensembles in the QED cavity. We have also shown a possibility of using additional quantum states (outside of the Hilbert subspace generated by the logical encoding) of atomic ensembles for multiple acceleration of some complex controlled operations, which are at the core of many quantum algorithms.

We have considered two types of two qubit gates: the one is based on nonequivalent Lamb shifts of collective transitions in the multi-atomic systems and the other one - on quantum transistor effect. The quantum transistor effect uses control of resonant transfer of excitation between target atomic ensembles due to the reversible change of atomic levels population. Quantum transistor effect is very promising for the construction of quantum computer especially taking into account comparatively long decoherence times of natural atoms. Both approaches can be realized by using existing facilities in quantum optics and in microwave QED technology.

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APPENDIX A.

Let's perform a unitary transformation of the Hamiltonian $H_s = e^{-s} H e^s$ that yields the following result in a second order of the perturbation theory:

$$(28) \quad H_s = H_0 + \frac{1}{2} [H_1, s],$$

when the relation

$$(29) \quad H_1 + [H_0, s] = 0,$$

holds.

Using the relation (29), we obtain

$$(30) \quad \begin{aligned} s = & \sum_{j_1} \left(\alpha_1^{(1)} g_{21}^{k_1} e^{i\vec{k}_1 \vec{r}_{j_1}} S_{21}^{j_1} a_{k_1} + \beta_1^{(1)} g_{21}^{k_1*} e^{-i\vec{k}_1 \vec{r}_{j_1}} S_{12}^{j_1} a_{k_1}^+ \right) \\ & + \sum_{j_2} \left(\alpha_2^{(1)} g_{21}^{k_1} e^{i\vec{k}_1 \vec{r}_{j_2}} S_{21}^{j_2} a_{k_1} + \beta_2^{(1)} g_{21}^{k_1*} e^{-i\vec{k}_1 \vec{r}_{j_2}} S_{12}^{j_2} a_{k_1}^+ \right) \\ & + \sum_{j_1} \left(\alpha_1^{(2)} g_{32}^{k_2} e^{i\vec{k}_2 \vec{r}_{j_1}} S_{32}^{j_1} a_{k_2} + \beta_1^{(2)} g_{32}^{k_2*} e^{-i\vec{k}_2 \vec{r}_{j_1}} S_{23}^{j_1} a_{k_2}^+ \right) \\ & + \sum_{j_2} \left(\alpha_2^{(2)} g_{32}^{k_2} e^{i\vec{k}_2 \vec{r}_{j_2}} S_{32}^{j_2} a_{k_2} + \beta_2^{(2)} g_{32}^{k_2*} e^{-i\vec{k}_2 \vec{r}_{j_2}} S_{23}^{j_2} a_{k_2}^+ \right) \\ & + \sum_{j_1} \left(\alpha_1^{(3)} g_{31}^{k_3} e^{i\vec{k}_3 \vec{r}_{j_1}} S_{31}^{j_1} a_{k_3} + \beta_1^{(3)} g_{31}^{k_3*} e^{-i\vec{k}_3 \vec{r}_{j_1}} S_{13}^{j_1} a_{k_3}^+ \right) \\ & + \sum_{j_2} \left(\alpha_1^{(3)} g_{31}^{k_3} e^{i\vec{k}_3 \vec{r}_{j_2}} S_{31}^{j_2} a_{k_3} + \beta_1^{(3)} g_{31}^{k_3*} e^{-i\vec{k}_3 \vec{r}_{j_2}} S_{13}^{j_2} a_{k_3}^+ \right), \end{aligned}$$

where

$$(31) \quad \alpha_{1,2}^{(1)} = -\beta_{1,2}^{(1)} = -\frac{1}{\varepsilon_{1,2}^{(2)} - \varepsilon_{1,2}^{(1)} - \hbar\omega_{k_3}},$$

$$(32) \quad \alpha_{1,2}^{(2)} = -\beta_{1,2}^{(2)} = -\frac{1}{\varepsilon_{1,2}^{(3)} - \varepsilon_{1,2}^{(2)} - \hbar\omega_{k_3}},$$

$$(33) \quad \alpha_{1,2}^{(3)} = -\beta_{1,2}^{(3)} = -\frac{1}{\varepsilon_{1,2}^{(3)} - \varepsilon_{1,2}^{(1)} - \hbar\omega_{k_3}}.$$

Substituting expressions (29) and (30) into (28), we get

(34)

$$\begin{aligned}
H_s = & \sum_{m=1,2} \frac{|g_{21}^{k_3}|^2}{\Delta_m^{(1)}} \sum_{i_m j_m} e^{i\vec{k}_1 \vec{r}_{i_m j_m}} S_{21}^{i_m} S_{12}^{j_m} + \sum_{m=1,2} \frac{|g_{21}^{k_3}|^2}{\Delta_m^{(1)}} \sum_{j_m} \left(S_{22}^{j_m} - S_{11}^{j_m} \right) n_{k_1} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(1)}} + \frac{1}{\Delta_2^{(1)}} \right) |g_{21}^{k_1}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{21}^{j_1} S_{12}^{j_2} + h.c. \right) \\
& + \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(1)}} \sum_{j_1} \left(g_{32}^{k_2} g_{21}^{k_1} e^{i(\vec{k}_1 + \vec{k}_2) \vec{r}_{j_1}} S_{31}^{i_1} a_{k_2} a_{k_1} + h.c. \right) \\
& + \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(1)}} \sum_{j_2} \left(g_{31}^{k_3} g_{21}^{k_1*} e^{i(\vec{k}_3 - \vec{k}_1) \vec{r}_{j_2}} S_{32}^{i_2} a_{k_3} a_{k_1}^+ + h.c. \right) \\
& + \sum_{m=1,2} \frac{|g_{32}^{k_2}|^2}{\Delta_m^{(2)}} \sum_{i_m j_m} e^{i\vec{k}_2 \vec{r}_{i_m j_m}} S_{32}^{i_m} S_{23}^{j_m} + \sum_{m=1,2} \frac{|g_{32}^{k_2}|^2}{\Delta_m^{(2)}} \sum_{j_m} \left(S_{33}^{i_m} - S_{22}^{i_m} \right) n_{k_2} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(2)}} + \frac{1}{\Delta_2^{(2)}} \right) |g_{32}^{k_3}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_2 \vec{r}_{j_1 j_2}} S_{32}^{j_1} S_{23}^{j_2} + h.c. \right) \\
& - \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(2)}} \sum_{j_1} \left(g_{32}^{k_2*} g_{31}^{k_3} e^{i(\vec{k}_3 - \vec{k}_2) \vec{r}_{j_1}} S_{21}^{i_1} a_{k_3} a_{k_2}^+ + h.c. \right) \\
& - \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(2)}} \sum_{j_2} \left(g_{32}^{k_2} g_{21}^{k_1} e^{i(\vec{k}_3 - \vec{k}_2) \vec{r}_{j_2}} S_{31}^{i_2} a_{k_1} a_{k_2} + h.c. \right) \\
& + \sum_{m=1,2} \frac{|g_{31}^{k_3}|^2}{\Delta_m^{(3)}} \sum_{i_m j_m} e^{i\vec{k}_3 \vec{r}_{i_m j_m}} S_{31}^{i_m} S_{13}^{j_m} + \sum_{m=1,2} \frac{|g_{31}^{k_3}|^2}{\Delta_m^{(3)}} \sum_{j_m} \left(S_{33}^{j_m} - S_{11}^{j_m} \right) n_{k_3} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(3)}} + \frac{1}{\Delta_2^{(3)}} \right) |g_{31}^{k_3}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_3 \vec{r}_{j_1 j_2}} S_{31}^{j_1} S_{13}^{j_2} + h.c. \right) \\
& - \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(3)}} \sum_{j_1} \left(g_{32}^{k_2} g_{31}^{k_3*} e^{i(\vec{k}_2 - \vec{k}_3) \vec{r}_{j_1}} S_{12}^{j_1} a_{k_2} a_{k_3}^+ + h.c. \right) \\
& - \frac{1}{2} \sum_{m=1,2} \frac{1}{\Delta_m^{(3)}} \sum_{j_1} \left(g_{21}^{k_1} g_{31}^{k_3*} e^{i(\vec{k}_1 - \vec{k}_3) \vec{r}_{j_1}} S_{23}^{j_1} a_{k_1} a_{k_3}^+ + h.c. \right),
\end{aligned}$$

where $\Delta_{1,2}^{(1)} = \varepsilon_{1,2}^{(2)} - \varepsilon_{1,2}^{(1)} - \hbar\omega_{k_1}$, $\Delta_{1,2}^{(2)} = \varepsilon_{1,2}^{(3)} - \varepsilon_{1,2}^{(2)} - \hbar\omega_{k_2}$, $\Delta_{1,2}^{(3)} = \varepsilon_{1,2}^{(3)} - \varepsilon_{1,2}^{(1)} - \hbar\omega_{k_3}$.

In the absence of real photons in the cavity $n_{k_1} = n_{k_2} = n_{k_3} = 0$, we get from (34) the following expression for effective ET Hamiltonian:

$$\begin{aligned}
(35) \quad H_s = & \sum_m^{1,2} \sum_{j_m \mu} \epsilon_m^\mu S_{\mu\mu}^{j_m} + \sum_{m=1,2} \frac{|g_{21}^{k_3}|^2}{\Delta_m^{(1)}} \sum_{i_m j_m} e^{i\vec{k}_1 \vec{r}_{i_m j_m}} S_{21}^{i_m} S_{12}^{j_m} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(1)}} + \frac{1}{\Delta_2^{(1)}} \right) |g_{21}^{k_1}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{21}^{j_1} S_{12}^{j_2} + e^{-i\vec{k}_1 \vec{r}_{j_1 j_2}} S_{12}^{j_1} S_{21}^{j_2} \right) \\
& + \sum_{m=1,2} \frac{|g_{32}^{k_2}|^2}{\Delta_m^{(2)}} \sum_{i_m j_m} e^{i\vec{k}_2 \vec{r}_{i_m j_m}} S_{32}^{i_m} S_{23}^{j_m} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(2)}} + \frac{1}{\Delta_2^{(2)}} \right) |g_{32}^{k_3}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_2 \vec{r}_{j_1 j_2}} S_{32}^{j_1} S_{23}^{j_2} + e^{-i\vec{k}_2 \vec{r}_{j_1 j_2}} S_{23}^{j_1} S_{32}^{j_2} \right) \\
& + \sum_{m=1,2} \frac{|g_{31}^{k_3}|^2}{\Delta_m^{(3)}} \sum_{i_m j_m} e^{i\vec{k}_3 \vec{r}_{i_m j_m}} S_{31}^{i_m} S_{13}^{j_m} \\
& + \frac{1}{2} \left(\frac{1}{\Delta_1^{(3)}} + \frac{1}{\Delta_2^{(3)}} \right) |g_{31}^{k_3}|^2 \sum_{j_1 j_2} \left(e^{i\vec{k}_3 \vec{r}_{j_1 j_2}} S_{31}^{j_1} S_{13}^{j_2} + e^{-i\vec{k}_3 \vec{r}_{j_1 j_2}} S_{13}^{j_1} S_{31}^{j_2} \right) .
\end{aligned}$$